A note on complex ℓ_2 solutions of linear systems of difference equations[1]

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Abstract

Sufficient conditions are given for the existence and the uniqueness of complex ℓ_2 solutions of a non-homogeneous system of linear difference equations and of two general classes of delay systems of linear difference equations. In some cases bounds of the established solutions are also given. As a consequence of the space ℓ_2 where we work, information can be obtained about the asymptotic behavior of the established solutions and, the asymptotic stability of the zero equilibrium point of the systems under consideration. The method we use is a functional-analytic one.

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1 Main results

In this short note we present the main results of a forthcoming paper. More precisely we present conditions in order to establish the existence and uniqueness of complex ℓ_2 solutions of the following
a) non-homogeneous linear system of difference equations

$$f(n+1) = A(n)f(n) + g(n), \quad n = 1, 2, ..., \tag{1.1}$$

where $f(n) = (f_1(n), ..., f_k(n))$, $g(n) = (g_1(n), ..., g_k(n))$ are elements of $\ell_2^k = \underbrace{\ell_2 \times ... \times \ell_2}_{k-times}$ and $A(n) = (a_{ij}(n))$ a $k \times k$ matrix of complex sequences, k a finite positive integer.

b) Delay, homogeneous, linear system of difference equations

$$f(n+1) = \Lambda(n)f(n) + \sum_{r=1}^{R} A_r(n)f(n-r), \quad n = 1, 2, ..., \tag{1.2}$$

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where $f(n) = (f_1(n), ..., f_k(n))$, $\Lambda(n) = (\lambda_{ij}(n))$ and $A_r(n) = (a_{ij}^{(r)}(n))$, $k \times k$ matrices of complex sequences, k a finite positive integer.

c) Delay, non-homogeneous, linear system of difference equations

$$(n - d_i) f_i(n - d_i + 1) = a_{i1}(n) f_1(n) + ... + a_{ik}(n) f_k(n) + g_i(n), \qquad (1.3)$$

where $a_{ij}(n)$, $n = 1, 2, ...$, $i, j = 1, ..., k$ bounded complex sequences, $g_i(n) \in \ell_2$, $i = 1, ..., k$ and d_i, $i = 1, ..., k$ non-negative integers, k a finite positive integer.

The asymptotic stability of the homogeneous ($g(n) \equiv 0$) system (1.1) was studied in [4] and [7]. It is known that the autonomous (i.e. when $A(n) \equiv A =$ constant matrix) homogeneous system (1.1) is asymptotically stable, if and only if the absolute values of all the eigenvalues of the matrix A are less than one. However, it is stated in [7] and a counterexample is given, that if $\gamma_i(n)$, $i = 1, ..., k$ are the eigenvalues of $A(n)$ and $\gamma = \sup_{n \geq 1} \max_{1 \leq i \leq k} |\gamma_i(n)|$, then the inequality $\gamma < 1$ does not imply the asymptotic stability of the homogeneous system (1.1). In this note we give sufficient conditions so that the zero equilibrium point of (1.1) to be asymptotically stable (see theorem 1.1 and remark 1.2). Also the asymptotic behavior of the solutions of system (1.1) with $A(n) = I + B(n)$, where I is the identity matrix, was studied in [3] for $n \in \{k, k + 1, ...\}$, $k \in \{0, 1, 2, ...\}$. Finally the bounded solutions of system (1.1) were studied, among other things, in [2] for $n \in \mathbb{Z}$.

In [1], an asymptotic representation as $n \to \infty$ of the real solutions of (1.2), with Λ a real diagonal matrix, was obtained. Also in [8], the asymptotic behavior of (1.2) was studied for

$$k = 2, \ \Lambda(n) = I - A, \ R = m, \ A_r(n) \equiv 0, \ \forall \, r = 1, ..., m - 1, \ A_R \equiv A,$$

where A a constant matrix and I the identity matrix.

Finally, system (1.3) is the discrete equivalent of the following linear system of differential equations:

$$z^D \frac{df(z)}{dz} = A(z) f(z) + g(z), \qquad (1.4)$$

where z^D stands for $(z^{d_1}, ..., z^{d_k})$, $A(z) = \{a_{ij}(z)\}$ a $k \times k$ matrix and $f(z) = (f_1(z), ..., f_k(z))$, $g(z) = (g_1(z), ..., g_k(z))$.

By use of a functional-analytic method, it was proved in [5], that if $A(z)$ is a bounded operator on $H_2(\Delta)$ and $k - d \geq 0$, where $d = TrD$, $D = diag(d_1, ..., d_k)$, then the system (1.4) has at least $k - d$ linearly independent solutions in $H_2(\Delta)^k$, where $H_2(\Delta)^k = \underbrace{H_2(\Delta) \times ... \times H_2(\Delta)}_{k-times}$ is the Hilbert space consisting of the

k-vector element of the form $f(z) = (f_1(z), ..., f_k(z))$ and $f_i(z)$, $i = 1, ..., k$ are elements of the Hilbert space $H_2(\Delta)$ of analytic functions defined by

$$H_2(\Delta) = \left\{ g(z) : \Delta \to \mathbb{C}, \; g(z) = \sum_{n=1}^{\infty} a_n z^{n-1}, \text{ with } \sum_{n=1}^{\infty} |a_n|^2 < +\infty \right\},$$

where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

It is shown in [5] that the system (1.4) is equivalent to an operator equation in an abstract separable Hilbert space H. This operator equation is also equivalent to the system (1.3). This is the reason that we characterize (1.3) as the discrete equivalent of (1.4). For (1.3) we prove a result analogous to the above mentioned result for (1.4) proved in [5] (see theorem 1.4). This technique, i.e. the derivation of a linear difference equation or a system of linear difference equations, equivalent to a linear differential equation or a system of linear differential equations, by use of an operator equation, can be considered as an "ideal discretization", since no errors (of any kind) are involved in the procedure.

The aim of this note is to give sufficient conditions so that the systems (1.1) and (1.2) to have a unique complex solution in ℓ_2^k and the system (1.3) to have some linearly independent solutions in ℓ_2^k, where $\ell_2^k = \underbrace{\ell_2 \times ... \times \ell_2}_{k-times}$ is the Hilbert space consisting of the k-vector element of the form $x(n) = (x_1(n), ..., x_k(n))$, where $x_i(n)$, $i = 1, ..., k$ is an element of the Hilbert space ℓ_2 defined by:

$$\ell_2 = \left\{ y(n) : \mathbb{N} \to \mathbb{C} \text{ with } \sum_{n=1}^{\infty} |y(n)|^2 < +\infty \right\}.$$

Once we have established, under certain conditions, a solution of (1.1) or (1.2) or (1.3) in ℓ_2^k, it is obvious from the definition of ℓ_2^k that $\lim_{n \to \infty} f_i(n) = 0$, $\forall$ $i = 1, ..., k$. Thus the zero equilibrium point $\underbrace{(0, ..., 0)}_{k-times}$ is an asymptotically stable equilibrium point of (1.1) or (1.2) or (1.3). More precisely our results are the following:

Theorem 1.1. Consider the system (1.1). If one of the following conditions hold:

$$\sup_i \sum_{j=1}^{k} \sup_n |a_{ij}(n)| < 1, \tag{1.5}$$

or

$$\lim_{n \to \infty} a_{ij}(n) = 0, \;\; \forall \; i, j = 1, ..., k, \tag{1.6}$$

or

$$\lim_{n\to\infty} a_{ij}(n) = \beta_{ij} < \infty, \ \text{ with } \ \sup_i \sum_{j=1}^{k} |\beta_{ij}| < 1, \ \forall \ i, j = 1, ..., k, \qquad (1.7)$$

then the system (1.1) has a unique solution in ℓ_2^k. Moreover when (1.5) holds, the solution $f(n)$ is bounded by

$$\|f(n)\|_{\ell_2^k} \leq \frac{\|g(n)\|_{\ell_2^k} + \|f(1)\|_{\ell_2^k}}{1 - \sup_i \sum_{j=1}^{k} \sup_n |a_{ij}(n)|}, \qquad (1.8)$$

where $f(1) = (f_1(1), ..., f_k(1))$.

Remark 1.2. a) For the homogeneous ($g(n) \equiv 0$), autonomous, diagonal ($A = diag(a_1, ..., a_k)$ = constant matrix) system (1.1), condition (1.5) becomes

$$\sup_s |a_s| < 1. \qquad (1.9)$$

Since the eigenvalues λ_s of the diagonal matrix A are its diagonal elements a_s, condition (1.9) becomes

$$\sup_s |\lambda_s| < 1,$$

from which we have $|\lambda_s| < 1, \forall \ s = 1, ..., k$. Thus a consequence of theorem 1.1 is that, if condition (1.9) holds, then the zero equilibrium point of the homogeneous, autonomous, diagonal system (1.1), is asymptotically stable, which is in accordance with the well-known result that the zero equilibrium point of the homogeneous, autonomous, diagonal system (1.1), is asymptotically stable if the absolute values of all the eigenvalues of A are less than one.

b) Theorem 1.1 gives sufficient conditions so that, the zero equilibrium point of system (1.1) to be asymptotically stable, which is a quite useful information as implied in [4] and [7].

Theorem 1.3. Consider the system (1.2). If one of the following conditions hold:

$$\sup_i \sum_{j=1}^{k} \sup_n |\lambda_{ij}(n)| + \sum_{r=1}^{R} \sup_i \sum_{j=1}^{k} \sup_n |a_{ij}^{(r)}(n)| < 1, \qquad (1.10)$$

or

$$\lim_{n\to\infty} \lambda_{ij}(n) = 0, \ \lim_{n\to\infty} a_{ij}^r(n) = 0, \ \forall \ i, j = 1, ..., k, \ r = 1, ..., R \qquad (1.11)$$

or

$$\lim_{n\to\infty} \lambda_{ij}(n) = \gamma_{ij}, \ \lim_{n\to\infty} a_{ij}^r(n) = \beta_{ij}^r, \ \text{ with } \ \sup_i \sum_{j=1}^{k} |\gamma_{ij}| + \sum_{r=1}^{R} \sup_i \sum_{j=1}^{k} |\beta_{ij}^{(r)}| < 1,$$

$$(1.12)$$

$\forall\, i, j = 1, ..., k,\, r = 1, ..., R$

then the system (1.2) has a unique solution in ℓ_2^k. Moreover if (1.10) holds, then the solution $f(n)$ is bounded by

$$\|f(n)\|_{\ell_2^k} \leq \frac{\|f(1)\|_{\ell_2^k}}{1 - \sup_i \sum_{j=1}^k \sup_n |\lambda_{ij}(n)| - \sum_{r=1}^R \sup_i \sum_{j=1}^k \sup_n |a_{ij}^{(r)}(n)|}, \quad (1.13)$$

where $f(1) = (f_1(1), ..., f_k(1))$.

Theorem 1.4. Consider the system (1.3). Let $D = diag(d_1, ..., d_k)$ and $TrD = d$. If $k - d \geq 0$, the system (1.3) has at least $k - d$ linearly independent solutions in ℓ_2^k.

The method we use in order to obtain our results is a functional-analytic one. It is actually a generalization of the method which was introduced in [6] and used recently in [9]-[11], for the study of linear and non-linear ordinary difference equations. This method is based on the representation of the space ℓ_2, by two shift operators of an abstract separable Hilbert space H.

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